Equilibria in first-price sealed-bid auctions of multiple heterogeneous objects: a numerical approach

Ludo Waltman Nees Jan van Eck

Econometric Institute
Erasmus School of Economics
Erasmus University Rotterdam
{lwaltman,nvaneck}@few.eur.nl

14th International Conference on
Computing in Economics and Finance

Paris, June 28, 2008

Outline of the presentation

- Introduction
- Model
- Methodology
- Analysis
- Conclusions

Introduction

- The problem of auctioning two non-identical and complementary objects is studied.
- The focus is on first-price sealed-bid auction mechanisms.
- Bayesian-Nash equilibria are approximated numerically.

Model

- There are two objects to be auctioned.
- Bidders have privately known values that are independently and identically distributed.
- x_1 and x_2 are drawn independently from a uniform distribution between 0 and 1.
- $x_{12} = (1 + \alpha)(x_1 + x_2)$, where $\alpha \geq 0$ has the same value for all bidders.
- Bidders are risk neutral.
- There are no budget constraints.
- Apart from bidders' values, all components of the model are commonly known to all bidders.

Auction mechanisms

Three first-price sealed-bid auction mechanisms are analyzed:

- Bundled first-price auction (BFPA)
- Simultaneous first-price auction (SFPA)
- Combinatorial first-price auction (CFPA)

Bundled first-price auction (BFPA)

The two objects are bundled together and auctioned in a single first-price auction.

- $b_{12} = \beta(x_{12})$

Simultaneous first-price auction (SFPA)

The two objects are auctioned separately in two simultaneous first-price auctions.

- $b_1 = \beta(x_1, x_2, \alpha)$
- $b_2 = \beta(x_2, x_1, \alpha)$

In case of synergies between the objects (i.e., $\alpha > 0$), this mechanism may suffer from the exposure problem.

Combinatorial first-price auction (CFPA)

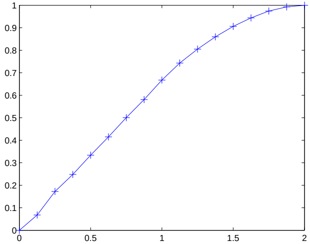
Bids are possible both on each object separately and on the two objects together. The bid or the pair of bids yielding the highest revenue is declared winning. The winning bidder(s) pay(s) its/their winning bid(s).

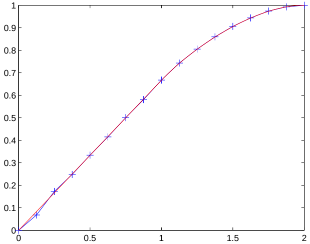
- $b_1 = \beta^{\mathrm{I}}(x_1, x_2, \alpha)$
- $b_2 = \beta^{\mathrm{I}}(x_2, x_1, \alpha)$
- $b_{12} = \beta^{\mathrm{II}}(x_1, x_2, \alpha) = \beta^{\mathrm{II}}(x_2, x_1, \alpha)$

Methodology (1)

Rather than analytically deriving a Nash equilibrium, equilibrium strategies are approximated numerically.

- Methodology is largely taken from O. Armantier, J.-P. Florens, and J.-F. Richard, Approximation of Bayesian Nash equilibria, submitted to *Journal of Applied Econometrics*.
- Example:





Methodology (2)

- Bidders use parametrized constrained strategies, e.g., piecewise linear strategies.
- Let n denote the number of bidders.
- Let the vector d_i denote the parameters of bidder i's constrained strategy.
- Let $\tilde{U}(d_i, d_{-i})$ denote bidder i's expected utility.
- A constrained strategy equilibrium can be found by solving the system of non-linear equations

$$\frac{\partial}{\partial d_i} \tilde{U}(d_i, d_{-i}) = \mathbf{0} \qquad \text{for } i = 1, \ldots, n. \tag{1}$$

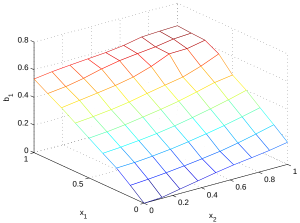
- Due to symmetry of the bidders, the constraint $d_1 = \ldots = d_n$ can be imposed.

Methodology (3)

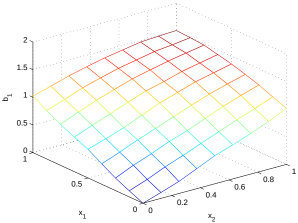
- $\tilde{U}(d_i, d_{-i})$ and its partial derivatives are difficult to evaluate analytically.
- $\tilde{U}(d_i, d_{-i})$ is therefore evaluated using Monte Carlo simulation, and its partial derivatives are numerically approximated.
- The system of non-linear equations is solved numerically using a MATLAB implementation of Powell's dogleg algorithm.

Analysis (1)

SFPA equilibrium bidding strategies:



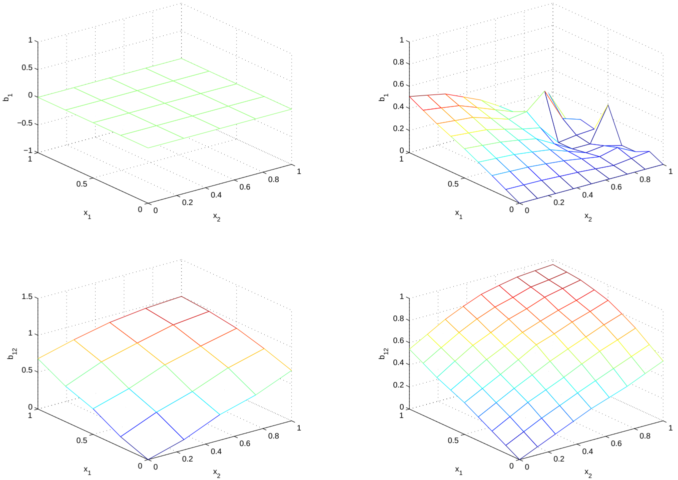
$n = 2, \alpha = 0.25$



$n = 2, \alpha = 2.00$

Analysis (2)

CFPA equilibrium bidding strategies ($n = 2, \alpha = 0.00$):



Equilibrium 1
(expected revenue = 0.765)

Equilibrium 2
(expected revenue = 0.718)

Analysis (3)

Expected revenue:

		$\alpha = 0.00$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.00$	$\alpha = 2.00$
$n = 2$	BFPA	*0.77*	*0.96*	*1.15*	*1.53*	*2.30*
	SFPA	*0.67*	0.91	1.13	1.53	2.29
	CFPA	0.72	0.94	1.15		
$n = 3$	BFPA	*1.00*	*1.25*	*1.50*	*2.00*	*3.00*
	SFPA	*1.00*	1.23	1.49	1.99	2.97
	CFPA	0.95	1.23	1.51		
$n = 4$	BFPA	*1.13*	*1.41*	*1.69*	*2.26*	*3.38*
	SFPA	*1.20*	1.42	1.68	2.24	3.35
	CFPA	1.15	1.45	1.70		
$n = 5$	BFPA	*1.21*	*1.52*	*1.82*	*2.43*	*3.64*
	SFPA	*1.33*	1.54	1.81	2.40	3.62
	CFPA	1.24	1.56	1.84		

- Numbers in *italics* were calculated analytically rather than numerically.
- Statistical uncertainty due to Monte Carlo simulation is negligible.

Analysis (4)

Efficiency:

		$\alpha = 0.00$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.00$	$\alpha = 2.00$
$n = 2$	BFPA	1.23	1.54	1.85	2.47	3.70
	SFPA	*1.33*	1.55	1.83	2.46	3.69
	CFPA	1.28	1.55	1.85		
$n = 3$	BFPA	1.35	1.69	2.03	2.70	4.05
	SFPA	*1.50*	1.70	2.00	2.69	4.04
	CFPA	1.44	1.71	2.03		
$n = 4$	BFPA	1.42	1.78	2.14	2.85	4.27
	SFPA	*1.60*	1.79	2.11	2.83	4.26
	CFPA	1.42	1.76	2.13		
$n = 5$	BFPA	1.48	1.85	2.22	2.95	4.43
	SFPA	*1.67*	1.86	2.19	2.94	4.42
	CFPA	1.62	1.87	2.22		

- Numbers in *italics* were calculated analytically rather than numerically.
- Statistical uncertainty due to Monte Carlo simulation is negligible.

Conclusions

- In the SFPA, if there is a strong synergy between the two objects, bidders circumvent the exposure problem by bidding almost equally on both objects.
- In the CFPA, there seem to be (at least) two Bayesian-Nash equilibria. In one equilibrium, the CFPA resembles the BFPA.
- In general, no auction mechanism seems to dominate the others in terms of expected revenue. The differences are usually quite small.
- Unfortunately, the accuracy of the numerical approximations is difficult to assess.
- For the CFPA, the accuracy of the numerical approximations seems questionable, in particular for larger n.